Main features of single particle quantum mechanics (QM)

1. Wave mechanics (1D)

In QM measurement always disturb the system in such a way that it is impossible to measure (with high accuracy) simultaneously all dynamical variables (observables). It means that in general case we can predict only probability to find the system in this or that state

The full information one can get from different measurements is contained in wave function $\Psi(q,t)$

It is postulated that this wave function obeys Shrödinger Eq.

(1) $$i\hbar \frac{\partial}{\partial t}\Psi(q,t) = \hat{H}\Psi(q,t)$$

where $\hat{H}$ is the Hamiltonian. $\hat{H}$ is the operator which corresponds in the classical limit to the total energy of the system $\hat{H} \Rightarrow E = T + V$

kinetic energy — potential energy

$\hat{H} = H(\hat{p}, \hat{q})$ $\hat{p}$ - momentum, $\hat{q}$ - coordinate

From experiments in atomic physics it is known that one can not measure simultaneously momentum and coordinate of a particle. The variances of momentum Δp and coordinate Δq always satisfy Heisenberg uncertainty relation

(2) $$\Delta p \Delta q \gtrsim \hbar$$

If for operators $\hat{p}$ and $\hat{q}$ we postulate the algebra

$$(3) \qquad [\hat{q}, \hat{p}] = i\hbar \;, \quad [\hat{p}, \hat{q}] = 0 \;, \quad [p, p] = 0$$

(the symbol $[\hat{a}, \hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a}$ is the <u>commutator</u> of $\hat{a}, \hat{b}$)
then we immediately get uncertainty relation (2).
One can satisfy (3) by choosing $\hat{p} = -i\hbar \frac{\partial}{\partial q}$, $\hat{q} = q$

$$[\hat{q}, \hat{p}] \Psi(q) = q\left(-i\hbar \frac{\partial}{\partial q}\right)\Psi - (-i\hbar)\frac{\partial}{\partial q}(q\Psi) =$$

$$= \cancel{-i\hbar q \partial_q \Psi} + i\hbar \Psi + \cancel{i\hbar q \partial_q \Psi} = i\hbar \Psi(q)$$

Since $\Psi(q)$ is arbitrary $\Longrightarrow$ commutation relation (3)

The Hamiltonian for a free particle in a static potential $V(q)$ takes the form

$$E_{cl}(p, q) = \frac{p^2}{2m} + V(q) \Longrightarrow \hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + V(q)$$

(There is problem of ordering operators $\hat{p}, \hat{q}$ if E_{cl} contains products pq !!!)

Physical interpretation of wave function:

$|\Psi(q,t)|^2 dq$ is the probability to find a particle with in interval $q, q+dq$

The probability description demands that wave function can be normalized. So physical states are those which satisfy

$$\int_{-\infty}^{\infty} |\Psi(q,t)|^2 dq < \infty$$

{ for continous spectrum Ψ-function is normalized on δ-function }

The above description is so called coordinate representation in Schrödinger picture

Representation of what? To answer this question we should formulate more general (abstract) description of quantum mechanics. It has been done 70 year ago by Dirac.

What is mathematical meaning of noncommuting objects like $\hat{p}, \hat{q}$? One can represent them as matrices (quadratic, infinite dimensional)
*(Actually it was the original formulation of QM by Heisenberg)

Then we can imagine the matrix with only one column $\begin{pmatrix} \vdots \end{pmatrix}$, which is nothing but vector. The Schröd. Eq. (1) is a linear equation, thus these vectors forms linear (vector) space.

The infinite dim. vector space of square integrable function (this property is needed for statistical interpretation) is known in mathematics as Hilbert space

Any quantum state can be represented as a vector $|\psi\rangle$ (state vector) in a Hilbert space (up to phase factor $e^{i\alpha}$) ↑ "ket vector"

As in any linear space one can choose some basis in a Hilbert space and determine the "coordinate" of state vector $|\psi\rangle$ with respect to this basis.

Different basises will give us different representations usually used in QM (coordinate, momentum, coherent state representation etc.)

It can be shown that eigenvectors of <u>any</u> Hermitian operator span the whole Hilbert space and thus they can be choosen as basis vectors.

"Coordinate representation"

$\hat{q}$ is Hermitian operator
position operator

$$\hat{q}|q\rangle = q|q\rangle$$

c-number

eigenvalues (spectrum) eigenvectors

The set of $\{|q\rangle\}$ is orthonormal

$$\langle q'|q\rangle = \delta(q-q')$$

and complete

$$\int dq\, |q\rangle\langle q| = I \tag{4}$$

unit operator

"Coordinates" of a state vector in this basis ~~...~~ are

$$\langle q|\Psi\rangle \equiv \Psi(q) \qquad |\Psi\rangle = \int dq\, \Psi(q)\,|q\rangle$$

scalar product

The momentum operator in coordinate representation looks like

$$\hat{p} = -i\hbar \frac{\partial}{\partial q} \tag{5}$$

(we have already checked that the commutation relation (3) is satisfied)

"Momentum representation"

Analogously one can use eigenvectors of momentum operator $\hat{p}$ as basis vectors in Hilbert space

$$\hat{p}|p\rangle = p|p\rangle \qquad \left\{ \begin{array}{l} \langle p'|p\rangle = \delta(p'-p) \\ \int dp\, |p\rangle\langle p| = I \end{array} \right.$$

c-number

"Coordinates" of state vector $|\Psi\rangle$ in momentum basis are

$\langle p|\Psi\rangle \equiv \Psi(p)$ — Ψ-function in "momentum representation"

In this basis momentum operator is (by definition) diagonal

$$\hat{p}|p\rangle = p|p\rangle$$

On contrary position operator $\hat{q}$ is nondiagonal. It takes the form

$$[\hat{q}, \hat{p}] = i\hbar \quad \Rightarrow \quad \hat{q} = \underset{\text{"plus" sign}}{+} i\hbar \frac{\partial}{\partial p} \qquad (6)$$

Let us find the overlap of two basises $\langle q|p\rangle$

$$\langle q|\hat{p}|p\rangle = p\langle q|p\rangle$$

$$\Downarrow$$

$$\int dq' \underbrace{\langle q|\hat{p}|q'\rangle}_{-i\hbar\frac{\partial}{\partial q'}\delta(q-q')}\langle q'|p\rangle = +i\hbar\frac{\partial}{\partial q}\langle q|p\rangle$$

$$\Rightarrow +i\hbar\frac{\partial}{\partial q}\langle q|p\rangle = p\langle q|p\rangle$$

$$\Downarrow$$

$$\langle q|p\rangle = C e^{\frac{i}{\hbar}qp}$$

Normalization constant C is found from Eq.

$$\langle q|\hat{I}|q'\rangle = \delta(q-q')$$

$$\hat{I} = \int dp' |p'\rangle\langle p'|$$

$$\int dp' \langle q|p'\rangle\langle p'|q'\rangle = \delta(q-q')$$

$$|C|^2 \int dp' e^{\frac{i}{\hbar}qp'} e^{-\frac{i}{\hbar}p'q'} = |C|^2 \int dp' e^{\frac{i}{\hbar}p'(q-q')} = \delta(q-q')$$

$\{$ Recall that $\int_{-\infty}^{\infty} dk\, e^{ikx} = 2\pi\delta(x)$ $\}$

$$\Downarrow$$

$$|C|^2 \hbar 2\pi\, \delta(q-q') = \delta(q-q')$$

$$C = \frac{e^{i\alpha}}{\sqrt{2\pi\hbar}} \quad \Rightarrow \quad \langle q|p\rangle = \frac{1}{(2\pi\hbar)^{1/2}} e^{\frac{i}{\hbar}pq}$$

$\Big\{$ For n-dimensional space one gets $\langle \vec{q}|\vec{p}\rangle = \frac{1}{(2\pi\hbar)^{n/2}} e^{\frac{i}{\hbar}\vec{p}\vec{q}}$ $\Big\}$

The transformation from momentum to coordinate basis

$$(7) \qquad \hat{1}|p\rangle = \int dq\, |q\rangle \underbrace{\langle q|p\rangle}_{\sim\, e^{\frac{i}{\hbar}pq}}$$

is nothing but Fourier transformation.

2. Schrödinger and Heisenberg representations of quantum evolution

State vector $|\Psi\rangle$ contains all information about system. It can be given by performing all compatable measurements on the system at some moment t_0.
How does $|\Psi\rangle$ evolves in time?
The answer — according Schr. equation

$$i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$$

If $H \neq H(t)$, this eq. can be solved formally

$$(8) \qquad |\Psi(t)\rangle = \underbrace{e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}}_{U(t,t_0)\text{ is the evolution operator}} |\Psi(t_0)\rangle$$

Properties:

$$(9) \qquad \begin{cases} UU^+ = U^+U = I & U \text{ is unitary operator (because } H^+ = H) \\ U(t_1,t_0)\,U(t_0,t_2) = U(t_1,t_2) & \end{cases}$$

For small time intervals $\Delta t = t - t_0 \to 0$

$$U(t,t_0) = I - \frac{i}{\hbar}\hat{H}\Delta t$$

(Hamiltonian is the generator of infenitesimal shifts in time)

In the above picture (Schrödinger representation) dynamical variables (operators in Hilbert space) do not depend on time. However their matrix elements depend on time due to the time-dependence of state vectors

$$\langle A_s\rangle = \langle\Psi_s(t)|\hat{A}_s|\Psi_s(t)\rangle \Rightarrow i\hbar\frac{d}{dt}\langle A_s\rangle = \langle\Psi_s|[A_s,H]|\Psi_s\rangle$$

~~Often~~ It is more convenient often (especially in MBT) to use another equivalent description of time evolution. If one defines the state vector ~~by~~ $|\psi_H\rangle$ by unitary transformation

for bra-vectors

$$|\psi_H\rangle = U^{-1}|\psi_S(t)\rangle \quad ; \quad \langle\psi_H| = \langle\psi_S(t)|U$$

$$(U^{-1})^{+} = U$$

then it will not depend on time $|\psi_H\rangle = |\psi_S(t_0)\rangle$.

Unitary transformation does not change the expectation values

$$\langle A\rangle = \underbrace{\langle\psi_S(t)|U}_{\langle\psi_H|}\,\underbrace{U^{-1}\hat{A}_S U}_{A_H(t)}\,\underbrace{U^{-1}|\psi_S(t)\rangle}_{|\psi_H\rangle} = \langle\psi_H|\hat{A}_H(t)|\psi_H\rangle$$

$$(10)\left\{\begin{aligned} |\psi_H\rangle &= U^{-1}|\psi_S(t)\rangle \\ \hat{A}_H(t) &= U^{-1}\hat{A}_S U \end{aligned}\right\}$$ Transformation from "S" to "H" pictures.

In Heisenberg representation operators depend on time explicitely

$$\frac{d}{dt}\hat{A}_H(t) = \underbrace{\left(\frac{d}{dt}U^{-1}\right)}_{+\frac{i}{\hbar}\hat{H}U^{-1}}\hat{A}_S U + U^{-1}\hat{A}_S\underbrace{\left(\frac{d}{dt}U\right)}_{-\frac{i}{\hbar}\hat{H}U} = \frac{i}{\hbar}\left(\hat{H}\hat{A}_H - \hat{A}_H\hat{H}\right) = \frac{i}{\hbar}\left[\hat{H}, A_H(t)\right]$$

$$(11)\quad \boxed{i\hbar\frac{d}{dt}\hat{A}_H(t) = \left[\hat{A}_H, \hat{H}\right]}$$ Heisenberg's equation

Heisenberg eq. plays fundamental role in MBT. We will use it quite often in what follows.

3. Harmonic Oscillator. Coherent States.

Before proceeding to many particle systems it is useful to consider one simple QM problem, namely, harmonic oscillator. We will look on this ~~very~~ well known ~~problem~~ and highly important problem from the side which will be useful for us in what follows.

To simplify the notation I'll restrict myself considering 1D case.

Hamiltonian: (particle in quadratic potential $V(q) \sim q^2$)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2 q^2 \qquad \psi(q,t) = \psi(q)e^{-\frac{i}{\hbar}Et}$$

Stationary Schr. Eq. (equation for spectrum)

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{m}{2}\omega^2 q^2\right)\psi(q) = E\psi(q)$$

Dimensionless coordinate $\qquad \xi = q/q_0 \qquad q_0 \equiv \sqrt{\frac{\hbar}{m\omega}}$

$$(12) \qquad \frac{\hbar\omega}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right)\psi(\xi) = E\psi(\xi)$$

It is useful to introduce operators

$$(13) \qquad \underbrace{a \equiv \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial\xi}\right)}_{\frac{m\omega\hat{q} + i\hat{p}}{\sqrt{2\hbar\omega m}}} \qquad \underbrace{a^+ = \frac{1}{\sqrt{2}}\left(\xi - \frac{\partial}{\partial\xi}\right)}_{\frac{m\omega\hat{q} - i\hat{p}}{\sqrt{2\hbar\omega m}}}$$

It is easy to verify that a, a^+ satisfy Heisenber-Weyl algebra

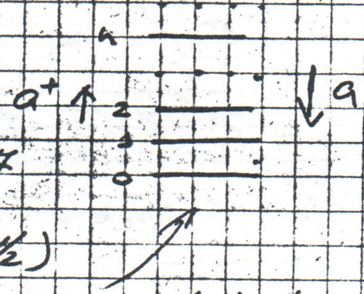
$$(14) \qquad \left\{ [a, a^+] = 1\ ;\ [a, a] = 0 \quad [a^+, a^+] = 0 \right\}$$

In terms a, a^+ Hamiltonian ~~(12)~~ takes simple form.

$$(15) \quad \hat{H} = \hbar\omega\left(a^+a + \tfrac{1}{2}\right)$$

If $|\psi_0\rangle$ is the eigenvector of $\hat{H}$ with energy E_0 then

$$\hat{H}\big(\underbrace{a^+|\psi_0\rangle}_{|\psi_1\rangle}\big) = (E_0 + \hbar\omega)\big(\underbrace{a^+|\psi_0\rangle}_{|\psi_1\rangle}\big)$$



$|\psi_1\rangle = a^+|\psi_0\rangle$ is also eigenvector with energy $E_0 + \hbar\omega$. ~~Then~~ The spectrum $E_n = \hbar\omega(n + 1/2)$

$n = 0, 1, 2, \dots$

equidistant set of levels

a^+ acts as creation operator

a — " — annihilation — " —

Ground state $|0\rangle$ satisfy eq. $\boxed{a|0\rangle = 0}$ (15)

Let us find g.s. wave function in x-representation

From Eq. (15) and definition (13) one finds

$$\left(\xi + \frac{d}{d\xi}\right)\psi_0(\xi) = 0 \Rightarrow \psi_0 \sim e^{-\xi^2/2}$$

$\Downarrow$ normalization

$$\int_{-\infty}^{\infty} d\xi\, \psi_0^*\psi_0 = 1$$

Analogously $\psi_n \sim (a^+)^n \psi_0$.

After normalization

$$\boxed{\psi_n = \frac{1}{\sqrt{n!}}(a^+)^n \psi_0}$$

Consider operator

$$\hat{n} = a^+ a \qquad \hat{n}|n\rangle = n|n\rangle \qquad n = 0, 1, 2, \ldots$$

It commutes with Hamiltonian $[H, \hat{n}] = 0$. One can interpret it as an operator of number of excitations

Notice that neither a^+ nor a are diagonal in n-representation

$$a^+|n\rangle \underset{\sqrt{n+1}}{\sim} |n+1\rangle \qquad a|n\rangle \underset{\sqrt{n}}{\sim} |n-1\rangle$$

$$a^+a|n\rangle = \sqrt{n}\, a^+|n-1\rangle = \sqrt{n}\sqrt{n}\,|n\rangle = n|n\rangle$$

<u>Coherent states</u>

Now let's put the question — (i) Is there representation in which a is diagonal?

(ii) If yes – what is the physical meaning of these states?

So we look for states $|\alpha\rangle$ such that

(16) $$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle}$$ Definition of coherent states

α — complex c-number (a is not Hermitian!!!)

If $|\alpha\rangle$ vector exists it can be represented as series on $|n\rangle$ states

$$|\alpha\rangle = \sum_{n=0}^{\infty} \langle n|\alpha\rangle |n\rangle$$

$I = \sum_{n=0} |n\rangle\langle n|$
$|n\rangle$ are the eigenvectors of hermitian operator $\hat{n}$

$$\Downarrow$$

$$|\alpha\rangle = \sum_{n=0} |n\rangle\langle n|\alpha\rangle$$

We want to calculate matrix element $\langle n|\alpha\rangle$. From Eq. (16)

$$\langle n|a|\alpha\rangle = \alpha\langle n|\alpha\rangle \qquad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\Downarrow \qquad \langle n|a = \sqrt{n+1}\,\langle n+1|$$

$$\sqrt{n+1}\,\langle n+1|\alpha\rangle = \alpha\langle n|\alpha\rangle$$

$$\langle n+1|\alpha\rangle = \frac{\alpha}{\sqrt{n+1}}\langle n|\alpha\rangle \xrightarrow{n+1\to n} \langle n|\alpha\rangle = \frac{\alpha}{\sqrt{n}}\langle n-1|\alpha\rangle =$$

$$= \frac{\alpha^2}{\sqrt{n(n-1)}}\langle n-2|\alpha\rangle = \ldots =$$

$$\ldots = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle$$

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \langle 0|\alpha\rangle = \text{?} \quad \left(\begin{matrix}\text{normalization}\\ \text{factor}\end{matrix}\right)$$

Since $\langle 0|\alpha\rangle$ does not depend on n it can be easily find from normalization condition

$$\langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \sum_{n,m=0}^{\infty}\frac{\alpha^n\alpha^{*m}}{\sqrt{m!\,n!}}\underbrace{\langle m|n\rangle}_{=\delta_{m,n}} = |\langle 0|\alpha\rangle|^2 \underbrace{\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}}_{e^{|\alpha|^2}} =$$

($|n\rangle$ form orthonormal set of states)

$$= |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2}$$

If we want $|\alpha\rangle$ states to be normalized $\langle\alpha|\alpha\rangle = 1$, then up to arbitrary phase factor

$$\langle 0|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}$$

$$\boxed{|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle} \qquad (17)$$

Decomposition of coherent states into $|n\rangle$ states.

By definition $a|\alpha\rangle = \alpha|\alpha\rangle$ then for bra vector we will have

$$\langle\alpha|a^+ = \alpha^*\langle\alpha|$$

$$\langle\alpha|a^+a|\alpha\rangle = |\alpha|^2 \equiv \bar{n}$$

average number of "particles" in coherent state

$|n\rangle$ states are the states with definite number of quanta n.
Coherent states $|\alpha\rangle$ are a mixture of $|n\rangle$ states, so it has no definite number of excitation quanta. The physical meaning of eigenvalue α is $\bar{n} \equiv |\alpha|^2$. It characterizes the average number of excitations.
We can rewrite Eq. (17)

$$(18) \quad |\alpha\rangle = \sum_{n=0}^{\infty} W_n |n\rangle \qquad P_n = |W_n|^2 = \underbrace{\frac{\bar{n}^n}{n!} e^{-\bar{n}}}_{\text{Poissonian distribution}}$$

The probability to find oscillator in state $|n\rangle$ obeys Poissonian distribution

Properties of coherent states:

(1) Coherent states minimize Heisenberg uncertainty relation

$$\Delta p_c \, \Delta q_c = \tfrac{1}{2}\hbar \qquad \Delta p^2 = \langle p^2\rangle - \langle p\rangle^2$$
$$\Delta q^2 = \langle q^2\rangle - \langle q\rangle^2$$

Roughly speaking, in c.s.r. a system (harmonic oscillator) behaves very much like classical one. What does it mean?

We know that for classical oscillator both coordinate and momentum are periodic function of time t with period $T = 2\pi/\omega$

$$(19) \quad \begin{aligned} x_{cl}(t) &\sim \cos(\omega t + \varphi_0) \\ p_{cl}(t) &\sim -\omega \sin(\omega t + \varphi_0) \end{aligned}$$

Let us find average values of ~~operator~~ position operator and momentum operator in coherent state representation

$$\left(a = \frac{m\omega q + ip}{\sqrt{2\hbar\omega m}} \qquad a^+ = \frac{m\omega q - ip}{\sqrt{2\hbar\omega m}} \right)$$

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\Downarrow$$

$$\frac{1}{\sqrt{2\hbar\omega m}}(m\omega q + ip)|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha = \alpha(t)$$

$$m\omega\langle q\rangle_c + i\langle p\rangle_c = \sqrt{2\hbar\omega m}\,\alpha = \qquad \left\{ \langle\alpha|\alpha\rangle = 1 \right\}$$

$$= \sqrt{2\hbar\omega m}\,(\mathrm{Re}\,\alpha + i\,\mathrm{Im}\,\alpha)$$

$$\langle q\rangle_c = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\alpha(t)$$

(20) $$\langle p\rangle_c = \sqrt{2\hbar\omega m}\,\mathrm{Im}\,\alpha(t)$$

We need to know how α depends on time. We know t-dependence of $|n\rangle$ state $\Rightarrow e^{-iE_n t/\hbar}$, $E_n = \hbar\omega(n+1/2)$

$$|\alpha(t)\rangle \sim \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega t(n+1/2)}|n\rangle = e^{-\frac{i\omega t}{2}}|\alpha e^{-i\omega t}\rangle$$

$|\alpha\rangle$ - vector rotates with varying time with frequency $\underline{\underline{\omega}}$

(21) $$\alpha(t) = \alpha e^{-i\omega t} = (\mathrm{Re}\,\alpha + i\,\mathrm{Im}\,\alpha)(\cos\omega t - i\sin\omega t)$$

$$|\alpha| e^{-i\omega t - i\varphi_0}$$

eigenvalue at time t eigenvalue at $t=0$

From Eqs. (20), (21) one gets

$$\langle q\rangle_c = \sqrt{\frac{2\hbar}{m\omega}}\left(\cancel{\mathrm{Re}\,\alpha\cos\omega t + \mathrm{Im}\,\alpha\sin\omega t}\;|\alpha|\cos(\omega t+\varphi_0)\right)$$

$$\langle p\rangle_c = -\sqrt{2\hbar\omega m}\,|\alpha|\sin(\omega t+\varphi_0)$$

$$q_0 = |\alpha|\sqrt{\frac{2\hbar}{\omega m}}$$

$$\left\{\begin{aligned} \langle q\rangle_c &= q_0\cos(\omega t+\varphi_0) \\ \langle p\rangle_c &= -q_0\,\omega m\sin(\omega t+\varphi_0) \end{aligned}\right\}$$

We can get the same result from another considerations.
In Heisenberg representation operators a, a^+ satisfy equations as follows

$$i\hbar \frac{d}{dt} a = [a, H] = \hbar\omega [a, a^+ a] = \hbar\omega (a a^+ a - a^+ a^2) =$$

$$H = \hbar\omega(a^+ a + \tfrac{1}{2})$$

$$= \hbar\omega (a a^+ - a^+ a) a = \hbar\omega a$$

$$[a, a^+] = 1$$

$$i\hbar \frac{d}{dt} a^+ = [a^+, H] = \hbar\omega [a^+, a^+ a] = \hbar\omega (a^{+2} a - a^+ a a^+) =$$

$$= \hbar\omega a^+ (a^+ a - a a^+) = -\hbar\omega a^+$$

$$[a^+, a] = -1$$

The integration of these differential eqs. is straightforward

$$a(t) = a_0 e^{-i\omega t} \qquad a^+(t) = a_0^+ e^{i\omega t}$$

From Eq. (13.)

$$\begin{cases} \hat{q}(t) = \sqrt{\frac{\hbar}{2m\omega}} (a^+ + a) = \sqrt{\frac{\hbar}{2m\omega}} \left(a_0^+ e^{i\omega t} + a_0 e^{-i\omega t} \right) \\ \hat{p}(t) = i m\omega \sqrt{\frac{\hbar}{2m\omega}} (a^+ - a) = i m\omega \sqrt{\frac{\hbar}{2m\omega}} \left(a_0^+ e^{i\omega t} - a_0 e^{-i\omega t} \right) \end{cases}$$

These two equations are quantum mechanical analog of second quantization representation of field operators in quantum field theory.
As $\langle\alpha|a|\alpha\rangle = \alpha$, $\langle\alpha|a^+|\alpha\rangle = \alpha^*$ we easily get above equations for $\langle\hat{q}\rangle_c$ and $\langle\hat{p}\rangle_c$

(ii) coherent states are nonorthogonal

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} < 1$$

they span the whole Hilbert space and can be used as a basis (actually the set of states $|\alpha\rangle$ is even overcomplete)

To prove the statement (ii) we have to learn how to work with coherent states. The idea is to use decomposition Eq. (17). It is useful to represent Eq. (17) in a slightly different form

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \underbrace{\frac{(a^+)^n}{\sqrt{n!}}|0\rangle}_{|n\rangle} = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha a^+)^n}{n!}|0\rangle =$$

$$= \underbrace{e^{-\frac{|\alpha|^2}{2}+\alpha a^+}}_{T(\alpha)}|0\rangle$$

Operator $T(\alpha)$ is not unitary. Define unitary operator

$$D(\alpha) = e^{\alpha a^+ - \alpha^* a} \qquad D^+(\alpha) = D^{-1}(\alpha)$$

and prove that $\quad |\alpha\rangle = D(\alpha)|0\rangle$

$$(17^a) \quad \boxed{|\alpha\rangle = e^{\alpha a^+ - \alpha^* a}|0\rangle} = D(\alpha)|0\rangle$$

Because a^+ and a do not commute we can not represent

$$e^{\alpha a + \beta a^+} \neq e^{\alpha a} e^{\beta a^+}$$

The correct formula is

$$(22)\quad \boxed{e^{\hat{A}}\cdot e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}} \quad \text{if} \quad \begin{matrix}[\hat{A},[\hat{A},\hat{B}]]=0\\ [\hat{B},[\hat{A},\hat{B}]]=0\end{matrix}$$

In our case $[a,a^+]=\underline{1}$ and the condition for Eq. (22) to be hold is satisfied. (For the proof of Eq. (22) see Appendix 1)

$$e^{\alpha a^+-\alpha^* a} = e^{-\frac{1}{2}[\alpha a^+,-\alpha^* a]}\, e^{\alpha a^+}\, e^{-\alpha^* a} = e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha a^+}\, e^{-\alpha^* a}$$

By definition of vacuum state it is annihilated by a

$$a|0\rangle = 0 \implies \underbrace{e^{-\alpha^* a}}_{1-\alpha^* a+\frac{(\alpha^*)^2}{2}a^2\dots}|0\rangle = |0\rangle$$

$$D(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha a^+}|0\rangle = T(\alpha)|0\rangle = |\alpha\rangle$$

Properties of D operators:

1. $D(\alpha)D(\beta) = e^{\alpha\beta^*-\alpha^*\beta}\, D(\beta)D(\alpha)$
2. $D(\alpha)D(\beta) = D(\alpha+\beta)\, e^{\frac{\alpha\beta^*-\alpha^*\beta}{2}}$

let us prove (2): $e^{\alpha a^+-\alpha^* a}\, e^{\beta a^+-\beta^* a} =$

$$= e^{\frac{1}{2}[\alpha a^+-\alpha^* a,\ \beta a^+-\beta^* a] + (\alpha+\beta)a^+ - (\alpha^*+\beta^*)a} = e^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)}\, D(\alpha+\beta)$$

$$\frac{1}{2}\Big\{-\alpha\beta^*\underbrace{[a^+,a]}_{-1} - \alpha^*\beta\underbrace{[a,a^+]}_{1}\Big\}$$

3. $[a, D(\alpha)] = \alpha D(\alpha)$

$[a^+, D(\alpha)] = \alpha^* D(\alpha)$

$\implies$

$D^+(\alpha)\, a\, D(\alpha) = a+\alpha$

$D^+(\alpha)\, a^+ D(\alpha) = a^+ + \alpha^*$

Statement (3) shows clearly that D acts as a shift operator in coherent state representation

(in x-representation the analogous operator is $T(a) = e^{\frac{i}{\hbar}\hat{p} a}$

$$T^{-1}(a) f(x) T(a) = f(x+a)$$)

With the help of Eq. (R^4) it is easy to prove statement (ii)

$$\langle\alpha|\beta\rangle = \langle 0|D^{-1}(\alpha) D(\beta)|0\rangle = e^{-\frac{|\alpha|^2+|\beta|^2}{2}} \langle 0|e^{\alpha^* a} e^{\beta a^+}|0\rangle =$$

$$= e^{-\frac{|\alpha|^2+|\beta|^2}{2}} \langle 0|e^{\beta a^+ + \alpha^* a + \frac{\alpha^*\beta}{2}\overbrace{[a,a^+]}^{=1}}|0\rangle =$$

$$\Downarrow$$

$$e^{\frac{\alpha^*\beta}{2}+\frac{\alpha^*\beta}{2}} e^{\beta a^+} e^{\alpha^* a}$$

$$= e^{-\frac{|\alpha|^2+|\beta|^2-2\alpha^*\beta}{2}} \underbrace{\langle 0|e^{+\beta a^+}}_{\langle 0|} \underbrace{e^{\alpha^* a}|0\rangle}_{|0\rangle}$$

$$\langle\beta|\alpha\rangle = e^{-\frac{|\alpha|^2+|\beta|^2-2\alpha\beta^*}{2}}$$

$$|\langle\alpha|\beta\rangle|^2 = e^{-(|\alpha|^2+|\beta|^2-\alpha^*\beta-\alpha\beta^*)} = e^{-|\alpha-\beta|^2} \leq 1.$$

To show the completeness of coherent states ~~we~~ it is sufficient to ~~prov~~ derive closure relation for states $|\alpha\rangle$. We can do it stright forwardly by calculating the integral

$$\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \sum_{n,m=0}^{\infty} \int_0^{\infty} d\varrho\,\varrho \int_0^{2\pi} d\varphi\, \frac{|n\rangle\langle m|}{\sqrt{m!\,n!}} \underbrace{e^{-\varrho^2}}_{(e^{-\frac{|\alpha|^2}{2}})^2} \varrho^{m+n} e^{-i(m-n)\varphi} =$$

$$\alpha = \varrho e^{i\varphi} \qquad d^2\alpha = \varrho\, d\varrho\, d\varphi$$

$$= \sum_{m,n=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{m!\,n!}} \int_0^{\infty} d\varrho\, \varrho^{m+n+1} e^{-\varrho^2} \underbrace{\int_0^{2\pi} d\varphi\, e^{-i(m-n)\varphi}}_{2\pi\delta(m-n)} =$$

$$\int_0^\infty dx\, x^k e^{-a^2x^2} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{2a^{k+1}}$$

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$$= 2\pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \underbrace{\int_0^\infty d\rho\, \rho^{2n+1} e^{-\rho^2}}_{\frac{n!}{2}} = \pi \underbrace{\sum_{n=0}^{\infty} |n\rangle\langle n|}_{\mathbb{1}} = \pi$$

(23) $$\boxed{\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \mathbb{1}}$$

Closure relation for coherent states

(Notice $|\alpha\rangle$ states are normalized to 1)

Any vector in Hilbert space can be expanded in coherent state basis

$$|\psi\rangle = \int \frac{d^2\alpha}{\pi} \langle\alpha|\psi\rangle |\alpha\rangle$$

$$\langle\alpha|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \langle n|\psi\rangle \equiv e^{-\frac{|\alpha|^2}{2}} \psi(\alpha^*)$$

holomorphic function.

$$\boxed{|\psi\rangle = \int \frac{d^2\alpha}{\pi} e^{-\frac{|\alpha|^2}{2}} \psi(\alpha^*) |\alpha\rangle}$$

Appendix 1.

If $[A,[A,B]]=0 \quad [B,[A,B]]=0$ then

$$e^{A}\cdot e^{B} = e^{A+B+\frac{1}{2}[A,B]}$$

Proof.

Let us consider operator which depends on parameter t

$$f(t) = e^{At}\, e^{Bt}$$

$$\frac{df}{dt} = Ae^{At}e^{Bt} + e^{At}B\underbrace{e^{Bt}}_{e^{-At}e^{At}} = \left(A + e^{At}Be^{-At}\right)f(t)$$

We show that $[B, e^{-At}] = e^{-At}\,C$

Then $e^{At}Be^{-At} = B + C$

$$[B, e^{-At}] = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}[B,A^n]$$

if $[B,[B,A]] = [A,[B,A]] = 0 \Rightarrow [B,A^n] = nA^{n-1}[B,A]$ (*)

It is true for $n=1 \xRightarrow{\text{trivial}} [B,A] = [B,A]$

$n=2 \quad [B,A^2] = \cancel{BA^2 - A} \; A[B,A] + [B,A]A \underset{[A,[A,B]]=0}{=}$

$= 2A[B,A]$

. . . .

Assume that Eq. (*) holds for n $\quad [B,A^n] = nA^{n-1}[B,A]$

~~Prove~~ Then prove that it is true for $n+1$

$$[B,A^{n+1}] = \cancel{A[B,A^n] + [B,A^n]A = nA^n[BA^n] + nA^{n-1}[BA]A} =$$

$$= (BA^n)A - A^{n+1}B = \left(A^nB + nA^{n-1}[B,A]\right)A - A^{n+1}B =$$

$$= A^nBA + nA^n[B,A] - A^n(AB) = A^n[B,A] + nA^n[B,A] =$$

$$= (n+1)A^n[B,A]$$ (as it should be for $k=n+1$)

So $\left[B, e^{-At}\right] = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} A^{n-1}[B,A] = -t\, e^{-At}[B,A]$.

$$e^{At} B e^{-At} = B - t[B,A] \Big\} + \frac{t^2}{2!}[A[A,B]] + \dots$$

if $[B[A,B]] \neq 0$

$$\frac{df}{dt} = (A + B - t[B,A]) f(t)$$

$$\Downarrow$$

$$\ln f(t) = (A+B)t + \frac{t^2}{2}[A,B]$$

$$e^{At} e^{Bt} = e^{(A+B)t + \frac{t^2}{2}[A,B]}$$

for $t = 1$

$$\boxed{e^{A} e^{B} = e^{A+B+\frac{1}{2}[A,B]}}$$

When proving the above operator equation we got important relation $[B, A^n] = n A^{n-1}[B,A]$ if $[B,A]$ is c-number. With the help of this equality it is easy to prove properties (3) for D-operators. For example

$$[a, D(\alpha)] = \alpha D(\alpha)$$

$$\left[a, e^{\alpha a^+ - \alpha^* a}\right] = e^{-\frac{|\alpha|^2}{2}} \left[a, e^{\alpha a^+} e^{-\alpha^* a}\right] =$$

$$= e^{-\frac{|\alpha|^2}{2}} \left(a e^{\alpha a^+} e^{-\alpha^* a} - e^{\alpha a^+} e^{-\alpha^* a} a \right) =$$

$$= e^{-\frac{|\alpha|^2}{2}} \underbrace{\left[a, e^{\alpha a^+}\right]}_{\alpha e^{\alpha a^+} [a, a^+]} e^{-\alpha^* a} = \alpha D(\alpha)$$

$$[a, a^+] = 1$$